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ANALYSIS OF THE TRANSPORTATION PROBLEM IN OBJECTIVES SPACE

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Abstract
The algorithm present a dynamic programming problem based on the destinations: at destination j, $2 \leq j \leq n$ only the associate costs can be used. The main problem of the algorithm is to find the polyhedrons vertices at the each iteration to result the efficiently solutions for the multiobjectives transportation problem.

***Keywords**: transportation problem, objectives space, polyhedron, vertex*

An approach based on dynamic programming is used to find an algebraic representation of a polyhedron in objectives space associate to a transportation problem with k linear objectives. This polyhedron has the same efficient structure just like the set of possible objectives values and, in addition, every its vertex is efficient.

The algebraic representation of this polyhedron has the following form $\left\{ y \in \mathbf{R}^k \,\middle|\, Hy \geq Ua + Vb \right\}$, where H, U, V are matrices independent of vector $\underline{a}$ (the availabilities vector) and vector $\underline{b}$ (the applications vector). The procedure is illustrated by a numerical example.

1. INTRODUCTION

The restrictions for the classical transportation problem of only one merchandise with m sources and n destinations are characterized by a quantity $a^{(i)}$ available in sources i, $i = \overline{1,m}$ and a quantity $b^{(j)}$ requisite to destination j, $j = \overline{1,n}$.

The problem is considered balanced (in equilibrium). We will consider $x_j^{(i)}$ the determination variable representing the transported quantity from source i, to destination j. The restrictions are:

$$\sum_{j=1}^{n} x_j^{(i)} = a^{(i)} \quad i = \overline{1,m} \tag{1}$$

$$\sum_{i=1}^{n} x_j^{(i)} = b^{(j)} \quad \mathrm{j} = \overline{1,n} \tag{2}$$

$$x_j^{(i)} \geq 0,\ i = \overline{1,m},\ \ j = \overline{1,n} \tag{3}$$

we used notation $a^{(i)}, b^{(j)}, x_j^{(i)}$ to introduce l index, $l = \overline{1,k}$, which represent the objective.

Often in a transportation problem, there are some objectives that enter in conflict, incommensurable, which have to be optimized according to restrictions (1), (2) and (3).

In sequel, is presented the case when exist $k > 1$ objectives which can be expressed as a linear combination of aggregate (assembly) variables.

Several authors ([2], [4], [5], [6]) have considered transportation problem in this general setting, and other authors ([1], [3], [7], [8]) have considered the problem in a setting where some linear objectives were not considered.

Certainly, when objectives are linear, multiobjective transportation problem represent a special case of general multiobjective linear programming problem.

(MOLP): Minimize $C \cdot x$ with restrictions $Ax = b,\ x \geq 0$, where C is the objectives $k \cdot q$ matrix, A is the restrictions $p \cdot q$ matrix, and $b \in \mathbf{R}^p$.

Despite this, as in the case of a singular objective transportation problem, the unique structure of transportation restrictions matrix, lead to algorithms for solving multiobjective transportation problem, algorithms which are more specialized than those developed for the general problem (MOLP).

Three such algorithms belong to Diaz [2], Gupta B. and Gupta R. [4] and Isermann [5]. All of them have the advantage of a special structure of transportation restrictions matrix and all imply a simplex analysis of restrictions set, and the algorithms given in [2] an [5] first enumerate the extreme Pareto optimal points, then identify all the optimal Pareto edges and faces of the restrictions set. Such a simplex analysis is a natural approach for solving multiobjectives linear programs and it is effective in many problems.

An alternative analysis approach of general multiobjectives linear problem was developed by Dauer and Saleh [9], [10]. In this approach they emphasize the analysis of objective set $y = C[X]$, more than the analysis of restrictions set $X = \left\{x \in \mathbf{R}^{\mathrm{q}} \,\middle|\, A \cdot x = b,\ x \geq 0\right\}$. The first advantage of analyzing y and not X is that the number k of objectives it is much less than q the number of variables, and y have less façade and extreme significant points than X. Moreover, from practical point of view, a decision factor is fundamental influenced by the considerations about the objectives space than the considerations about the restrictions space.

Therefore, an analysis of the objective space of linear multiobjectives programs has the advantage to be more easily and to provide a much better understanding than an analysis of restrictions space.

To achieve a complete analysis of objectives space is necessary an algebraic representation for the set y. Such a representation was developed for (MOLP) by Dauer and Saleh [9] and later was modified in [10] to achieve an algebraic representation for the polyhedron $\tilde{y} = y + \mathbf{R}_+^k$.

We notice that $\tilde{y}$ have the same efficient structure like y, and in addition have the properties that all his extreme points are efficacy. In the present problem, the special structure of transportation restrictions matrix is used to achieve an alternative variant of algebraic representation of polyhedron $\tilde{y}$ associate to a multiobjectives transportation problem. In particularly, the matrices H, U, V are constructed such that $\tilde{y}$ have the representation $\tilde{y} = \left\{ y \in \mathbf{R}^k \,\middle|\, Hy \geq Ua + Vb \right\}$, where $a = \left[a^{(1)}, \ldots, a^{(m)} \right]^{\mathrm{T}}$ and $b = \left[b^{(1)}, \ldots, b^{(n)} \right]^{\mathrm{T}}$.

The construction technique for the matrices H, U and V is achieved in an iterative manner using a forward dynamic programming approach who require $n-1$ iterations. The expression "forward dynamic programming" is justified by the reason that the iteration j, $j = \overline{2,n}$ depends just on the costs associate to destinations $1, 2, \ldots, j$. A consequence of this fact is that the matrices H, U and V constructed by the algorithm are completely independent of vector $\underline{a}$ and vector $\underline{b}$. Other consequences used in construction of H, U and V are discussed in Observation 6.

In sequel, at section 2 are introduced the main notations and is given an outline of construction algorithm for H, U and V.

In section 3 is shown that the algorithm from section 2 determine, in fact, the desired representation for $\tilde{y}$. In section 4, is given a numerical example to illustrate the algorithm, and in sequence 5 are given some conclusive observations.

2. THE ALGORITHM OUTLINE

The purpose of this section is to outline (sketch) the algorithm used in construction of the matrices H, U and V. We start to specify the used vectorial notations and the notations relating to the structure of the transportation problem that we use here.

- The r-th component of the column vector v is $v^{(r)}$, and v^T denoted the transpose of v.

- Given two vectors v_1 and v_2 we have: $\mathrm{v}_1 \leq \mathrm{v}_2$ if $\mathrm{v}_1^{(\mathrm{r})} \leq \mathrm{v}_2^{(\mathrm{r})}$, $(\forall) r$ and we have $\mathrm{v}_1 <\neq \mathrm{v}_2$ if $\mathrm{v}_1 \leq \mathrm{v}_2$ and $\mathrm{v}_1 \neq \mathrm{v}_2$.

- The symbol e is used to note the vector with all component equal with 1, i.e. $e^r = 1\ (\forall) r$. The number of e's component is that number who make the expression valid. E.g., to indicate $\sum_{i=1}^{m} a^{(i)} = \sum_{j=1}^{h} b^{(j)}$ we may write $e^{\mathrm{T}} \cdot a = e^{\mathrm{T}} \cdot b$. Analogue, the symbol O is used to note the line or column null vector or the null matrix.
- The symbol x is used to note the decisions variables vector for the transportation problem.

By convention, x is partitioned as $x^{\mathrm{T}} = \left[x_1^{\mathrm{T}} : x_2^{\mathrm{T}} : \ldots : x_n^{\mathrm{T}} \right]$, with ${x_j}^{\mathrm{T}} = \left[x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(m)} \right]$, $j = \overline{1, n}$ i.e. x_j is the decision vector associate to with destination j. The transportation restrictions matrix is noted with A and has the dimension $(m+n) \times m \cdot n$ and is partitioned like this $A = \begin{bmatrix} I & \vdots & I & \vdots & \cdots & \vdots & I \\ -Q_1 & \vdots & -Q_2 & \vdots & \cdots & \vdots & -Q_N \end{bmatrix}$, where I is the unit matrix with $m \times n$ rows and columns and Q_j is the matrix with all elements 1 on j-th line and O in rest ($m \times n$ rows and columns).
- The b letter is used to note the availabilities and applications vector and it is partitioned like this: $b^{\mathrm{T}} = \left[a^{\mathrm{T}} \vdots -d^{\mathrm{T}} \right]$, with $a^{\mathrm{T}} = \left[a^{(1)}, \ldots, , a^{(m)} \right]$ and $d^{\mathrm{T}} = \left[d^{(1)}, \ldots, d^{(n)} \right]$. At last, the letter C is used to note the objective matrix of dimension $k \times (m \cdot n)$ and it is partitioned like this $C = \left[C_1 \vdots C_2 \vdots \ldots C_n \right]$, with C_j the "cost" matrix associate with destination j, $j = \overline{1, n}$ of dimension $k \times m$.

With this notations, the linear multiobjectives transportation problem (MOTP) can be expressed as follow:

(MOTP): Minimize Cx corresponding to $x \in X$, with $X = \left\{ x \in \mathbf{R}^{mn} \middle| Ax = b,\ x \geq 0 \right\}$.

Next, we will be using both representations of X. Because the central object is the objective variables set, is considered $Y := C[x] = \left\{ y \in Y \middle| (\exists) x \in X \text{ so that } \mathrm{y} = \mathrm{C} \cdot \mathrm{x} \right\}$ and is noted with $E(y)$ the set all efficient point of Y, i.e. $E(y) := \left\{ y \in Y \text{ does not exist } \overline{\mathrm{y}} \in \mathrm{Y} \text{ with } \overline{\mathrm{y}} <\neq \mathrm{y} \right\}$.
- We define $\tilde{y} := Y + \mathbf{R}_+^k = \left\{ y + z \middle| y \in Y,\ z \in \mathbf{R}^k,\ z \geq 0 \right\}$ and observe that:

a) $E(\tilde{y}) = E(y)$;

b) any extreme point of $\tilde{y}$ is efficient.

These two remarks suggest that the achieving of a linear inequalities system to determine $\tilde{y}$ may be more useful in (MOTP) analysis than the achieving of a system to determine Y.

- In this moment, is sketched an construction algorithm for matrix H, U and V such that $\tilde{y} = \left\{ y \in \mathbf{R}^k \,\middle|\, Hy \geq U \cdot a + V \cdot b \right\}$;

- It is emphasizing that the algorithm require $e^{\mathrm{T}} \cdot a = e^{\mathrm{T}} \cdot b$. The case $e^{\mathrm{T}} \cdot a > e^{\mathrm{T}} \cdot b$ will be discussed in Observation 3.6 (c).

THE MAIN IDEA OF THE ALGORITHM

We defined iterative a sequence of $n-1$ polyhedrons $\psi_2, ..., \psi_n$, where ψ_2 depends on the matrices C_1 and C_2, and for $j = \overline{3, n}$, ψ_j depends on the matrices C_1 and C_j and on the vertices of the polyhedrons ψ_{j-1}. For all $j = \overline{2, n}$, the vertices of ψ_j are used to define the matrices $H^{[j]}, U^{[i]}, V^{[j]}$ such that the desirable matrices H, U and V that describe $\tilde{Y}$ are $H^{[j]}, U^{[i]}$ and $V^{[j]}$.

Before we precisely define the notations used in algorithm, we notice the fact that the main calculus effort is the enumeration of the vertices of the polyhedron ψ_j, $j = \overline{2, n}$.

There are known various methods for finding all vertices of convex polyhedral sets. In sections 4 and 5 will be discussed the calculus aspects of the algorithm.

In sequel, is presented the notations used in this algorithm.

I. The definition of the polyhedron ψ_j, $j = \overline{2, n}$:

Let Dj, $j = \overline{2, n}$, a nonnegative $q \times m$ matrix (the precise definition will be given in VII, and dimension q_j is explicated in III). The polyhedron ψ_j associated to D_j is defined like this:

$$\psi_j := \left\{ \left[h^{\mathrm{T}}, u^{\mathrm{T}}, \beta \right] \in \mathbf{R}^{q_j + m + 1} \,\middle|\, h^{\mathrm{T}} \cdot D_j + u^{\mathrm{T}} \geq \beta \cdot e^{\mathrm{T}}, h^{\mathrm{T}} \cdot e = 1, \left[h^{\mathrm{T}}, u^{\mathrm{T}}, \beta \right] \geq 0 \right\} \tag{4}$$

II. The definition of the matrix $\left[H_j : U_j : \mathrm{v}_j \right]$, $j = \overline{2, n}$:

Let $\left[h_1^{\mathrm{T}}, u_1^{\mathrm{T}}, \beta_1 \right], ..., \left[h_{r_j}^{\mathrm{T}}, u_{r_j}^{\mathrm{T}}, \beta_j \right]$ the vertices of the polyhedron ψ_j. The matrix $\left[H_j : U_j : \mathrm{v}_j \right]$ is defined as a $r_j \times \left(q_j + m + 1 \right)$ matrix which has as r-th row $\left[h_r^{\mathrm{T}}, u_r^{\mathrm{T}}, \beta_r \right]$. We notice that $\left[H_j : U_j : \mathrm{v}_j \right]$ is a nonnegative matrix and each of his rows has at most $m+1$ positive elements. In addition, because the matrix rows $\left[H_j : U_j : \mathrm{v}_j \right]$ are vectors from ψ_j, we have:

$$H_j \cdot D_j \geq -U_j + \mathrm{v}_j \cdot e^{\mathrm{T}}, \quad j = \overline{2, n} \tag{5}$$

III. The definition of the matrices $H^{[i]}$ and $U^{[j]}$, $j = \overline{1, n}$.

We remind that D_j is a $q_j \times m$ matrix, which will be defined in VII. Now, is sufficiently to say that $q_j = r_{j-1}$, for $j = \overline{2,n}$, with $r_1 = k$. Taking all these into account, we give the next definitions:

The $r_j \times k$ matrix $H^{[i]}$ is defined as:

$$H^{[i]} = \begin{cases} I, \ j=1 \\ H_j \cdot U^{[J-1]} \ j = \overline{2,n} \end{cases} \tag{6}$$

with I is the $k \times k$ unit matrix.

The $r_j \times m$ matrix $U^{[j]}$ is defined as:

$$U^{[j]} = \begin{cases} C_1 \ \ j=1 \\ H_j \cdot U^{[J-1]} \text{-} U_j \ \ j = \overline{2,n} \end{cases} \tag{7}$$

IV. The definition of the $r_p \times m$ matrix $C_j^{[p]}$, $j = \overline{2,n}$, $p = \overline{1, j-1}$.

$$C_j^{[p]} = \begin{cases} C_j - C_1, \ p = 1 \text{ şi j=}\overline{\text{2,n}} \\ H_p \cdot C_j^{[p-1]} + U_p, \ p = \overline{2, j-1}, \text{ j=}\overline{\text{3,n}} \end{cases} \tag{8a}$$

if and only if we have:

$$C_j^{[p]} = H^{[p]} \cdot C_j - U^{[p]}, \ p = \overline{1, j-1}, \ j = \overline{2,n} \tag{8b}$$

V. The definition of the vector c_j, $j = \overline{2,n}$.

c_j is defined as the infimum of the column vectors of $c_j^{[j-1]}$, with the infimum is considered with respect to natural order from Euclidean r_{j-1}–space, i.e. $c_j \le c \ (\forall) c$ the column vector of $C_j^{[j-1]}$, and if $\bar{c} \le c \ (\forall) c$ then $\bar{c} \le c_j$.

VI. The definition of the $r_j \times j$ matrix $V^{[j]}$, $j = \overline{1,n}$.

$$V^{[j]} := \begin{cases} O; \ j=1 \\ \left[H_j \cdot V^{[j-1]} : H_j \cdot c_j + v_j \right], \ j = \overline{2,n} \end{cases} \tag{9}$$

with O the null vector from $\mathbf{R}^k$.

VII. The definition of the $r_j \times m$ matrix D_j, $j = \overline{2,n}$.

$$D_j : C_j^{[j-1]} - c_j \cdot e^{\mathrm{T}} \tag{10}$$

Taking into account the definition of c_j, we notice that D_j is a nonnegative matrix. No negativity of the matrix D_j is not essentially for the algorithm, but is appropriate when we calculate vertices of polyhedron ψ_j.

3. THE REPRESENTATION OF $\tilde{Y}$

In this section, we establish that $\tilde{Y}$ have the representation: $\tilde{Y} = \left\{ y \in \mathbf{R}^k \left| H \cdot y \ge U \cdot a + V \cdot b \right. \right\}$, with $H = H^{[n]}$, $U = U^{[n]}$ and $V = V^{[n]}$.

Proposition 1 If $y \in \tilde{Y}$, then y verify $Hy \ge U \cdot a + V \cdot b$.

Prove. Since $y \in \tilde{Y}$, we have: $(\exists)\ x \in X$, such that $y \ge C \cdot x$. The objective vector $C \cdot x$ may be written as follow:

$$Cx = \sum_{j=1}^{n} C_j \cdot x_j + C_1 \cdot a + \sum_{j=2}^{n} \left(C_j - C_1 \right) \cdot x_j = C_1 \cdot a + \sum_{j=2}^{n} C_j^{[1]} \cdot x_j .$$

Thus, using notations $H^{[1]} = I$, $U^{[1]} = C_1$ and $V^{[1]} = O$, we have:

$$H^{[1]} \cdot y \ge U^{[1]} \cdot a + V^{[1]} \cdot b^{[1]} + \sum_{j=2}^{n} c_j^{[1]} \cdot x_j \tag{11}$$

Also, we have:

$$C_2^{[1]} \cdot x_2 = \left(C_2^{[1]} - c_2 \cdot e^{\mathrm{T}} \right) \cdot x_2 + c_2 e^{\mathrm{T}} \cdot x_2 = D_2 x_2 + c_2 d^{(2)} \tag{12}$$

In addition, using (5) for $j = 2$, we have:

$$\left(H_2 D_2 \right) \cdot x_2 \ge \left(-U_2 + \mathrm{v}_2 \cdot e^{\mathrm{T}} \right) \cdot x_2 =$$

$$= -U_2 x_2 + \mathrm{v}_2 \cdot d^{(2)} \ge -U_2 a + \mathrm{v}_2 \mathrm{d}^{(2)} + \sum_{\mathrm{j}=3}^{\mathrm{n}} \mathrm{U}_2 x_j \tag{13}$$

and, from the last inequality we deduce that $U_2 x_1 \ge O$.

Using (11), (12) and (13) we have:

$$H_2 \cdot H^{[1]} \cdot y \ge \left(H_2 U^{[1]} - U_2 \right) \cdot a + \left[H_2 V^{[1]} : H_2 c_2 + \mathrm{v}_2 \right] \cdot \left[d^{(1)}, d^{(2)} \right]^{\mathrm{T}} + \sum_{j=3}^{n} \left(H_2 \cdot C_j^{[1]} + U_2 \right) x_j$$

Now, with notation given in (6), (7), (8a) and (9), we have:

$$H^{[2]} \cdot y \ge U^{[2]} \cdot a + V^{[2]} \cdot \left[b^{(1)}, b^{(2)} \right]^{\mathrm{T}} + \sum_{j=3}^{n} C_j^{[2]} \cdot x_j .$$

Repeating this reasoning, we have:

$H^{[n]} \cdot y \ge U^{[n]} \cdot a + V^{[n]} \cdot b$ and hence $H \cdot y \ge U \cdot a + V \cdot b$.

In proposition 4 we give a reciprocal of proposition 1.

First, it is useful to prove next lemma and to introduce a new definition.

Lemma 2 Suppose that $y \in \mathbf{R}^k$ and the system $\left[h^{\mathrm{T}}, g^{\mathrm{T}} \right] \cdot \begin{bmatrix} C \\ A \end{bmatrix} \ge O$, $\left[h^{\mathrm{T}}, g^{\mathrm{T}} \right] \cdot \begin{bmatrix} y \\ b \end{bmatrix} < O$ $\left[h^{\mathrm{T}}, g^{\mathrm{T}} \right] \ge O$, $h \in \mathbf{R}^k$, $g \in \mathbf{R}^{m+n}$ have a solution $\left[h_0^{\mathrm{T}}, g_0^{\mathrm{T}} \right]$. Then $h_0 \ne 0$.

Prove. Suppose that $h_0 = O$. Then the system $g^{\mathrm{T}} \cdot A \geq O$, $g^{\mathrm{T}} \cdot b < O$, $g \geq O$, $g \in \mathbf{R}^{m+n}$, have a solution. Therefore, the system $\left[g^T, z^{\mathrm{T}}\right] \cdot \begin{bmatrix} A \\ -I \end{bmatrix} = O$, $\left[g^T, z^{\mathrm{T}}\right] \cdot \begin{bmatrix} b \\ O \end{bmatrix} < O$, $\left[g^T, z^{\mathrm{T}}\right] \geq O$, $g \in \mathbf{R}^{\mathrm{m+n}}$, $z \in \mathbf{R}^{m \cdot n}$ have a solution. Thus, according to Gale's theorem of linear inequalities alternative, the system $\begin{bmatrix} A \\ -I \end{bmatrix} \cdot x \leq \begin{bmatrix} b \\ O \end{bmatrix}, x \in \mathbf{R}^{mn}$ don't have a solution, and therefore $X = \phi$. Contradiction.

VIII. The definition of the polyhedron Φ_j, $j = \overline{2,n}$.

The polyhedron Φ_j associate with the matrix D_j is defined through:

$$\Phi_j := \left\{ \left[h^{\mathrm{T}}, u^{\mathrm{T}}, \alpha, \beta\right] \in \mathbf{R}^{y_{j-1}+m+2} \middle| h^{\mathrm{T}} \cdot D_j + u^{\mathrm{T}} + \alpha \cdot e^{\mathrm{T}} \geq \beta \cdot e^{\mathrm{T}}, \right. \left. h^{\mathrm{T}} \cdot e^{\mathrm{T}} = 1, \left[h^{\mathrm{T}}, u^{\mathrm{T}}, \alpha, \beta\right] \geq 0 \right\} \quad (14)$$

In the following observations, we emphasize some properties of Φ_j and his relations with Ψ_j.

Observation 3

a) If $\left[h^{\mathrm{T}}, u^{\mathrm{T}}, \alpha, \beta\right]$ is a vertex of Φ_j, then $\alpha = 0$. To prove this fact, we observe that if $\alpha > 0$, then we can write:

$$\left[h^{\mathrm{T}}, u^{\mathrm{T}}, \alpha, \beta\right] = \begin{cases} \frac{1}{2}\left[h^{\mathrm{T}}, u^{\mathrm{T}}, \alpha - \beta, 0\right] + \frac{1}{2}\left[h^{\mathrm{T}}, u^{\mathrm{T}}, \alpha + \beta, 2\beta\right], \text{ dacă } \alpha > \beta \\ \frac{1}{2}\left[h^{\mathrm{T}}, u^{\mathrm{T}}, 0, \beta - \alpha\right] + \frac{1}{2}\left[h^{\mathrm{T}}, u^{\mathrm{T}}, 2\alpha, \alpha + \beta\right], \text{ dacă } 0 < \alpha \leq \beta \end{cases}.$$

b) There is a bijection between the vertices of Φ_j and Ψ_j defined in (4). In particular, $\left[h^{\mathrm{T}}, u^{\mathrm{T}}, 0, \beta\right]$ is a vertex of Φ_j if and only if $\left[h^{\mathrm{T}}, u^{\mathrm{T}}, \beta\right]$ is a vertex of Ψ_j.

c) The extreme radius of the polyhedron Φ_j, $j = \overline{2,n}$, are the rows of the $(m+3) \times (r_{j-1} + m + 2)$ matrix

$$E_j = \begin{bmatrix} O & I & O & O \\ O & e^{\mathrm{T}}O & I & O \\ O & O & I & I \\ O & O & I & O \end{bmatrix}.$$

d) If $\left[h^{\mathrm{T}}, u^{\mathrm{T}}, \alpha, \beta\right] \in \Phi_j$, then he written as a convex combination of vertices of Φ_j plus a nonnegative linear combination of extreme radius of Φ_j. Thus, according to the previous observation and the definition of the matrix $\left[H_j, U_j, \mathrm{v}_j\right]$, there exist $z \in \mathbf{R}^{r_j}$, $w \in \mathbf{R}^m$ and $\gamma, \delta, \eta \in \mathbf{R}$ with $z^{\mathrm{T}} \cdot e = 1$.

$\left[z^{\mathrm{T}}, w^{\mathrm{T}}, \gamma, \delta, \eta\right] \geq O$ and

$$\left[h^{\mathrm{T}}, u^{\mathrm{T}}, \alpha, \beta\right] = \mathrm{z}^{\mathrm{T}} \cdot \left[H_j : U_j : 0 : \mathrm{v}_j\right] + \left[w^{\mathrm{T}}, \gamma, \delta, \eta\right] \cdot E_j \tag{15}$$

In sequel, we prove the reciprocal of the proposition 1.

Proposition 4 If $y \in \mathbf{R}^k$ satisfied $Hy \geq U \cdot a + V \cdot b$, then $y \in \tilde{Y}$.

Prove. Is proving that exist a $x \in X$ such that $y \geq C \cdot x$, i.e. the system $\begin{bmatrix} C \\ A \\ -I \end{bmatrix} \cdot x \leq \begin{bmatrix} y \\ b \\ 0 \end{bmatrix}$, $x \in \mathbf{R}^{mn}$ has a solution.

Let suppose the contrary. Then according to Gale's theorem, is shown that exist $h_0 \in \mathbf{R}^k$ and $g_0 \in \mathbf{R}^{m+n}$ such that

$$\left[h_0^{\mathrm{T}}, g_0^{\mathrm{T}}\right] \geq 0$$

$$h_0^{\mathrm{T}} \cdot C + g_0^{\mathrm{T}} \cdot A \geq 0 \tag{16}$$

$$h_0^{\mathrm{T}} y + g_0^{\mathrm{T}} \cdot b < 0 \tag{17}$$

According to lemma 2, we may suppose, without loosing generality, that $h_0^{\mathrm{T}} \cdot e = 1$.

Let $g_0^{\mathrm{T}} := \left[u_0, \mathrm{v}_0^{\mathrm{T}}\right]$, where $u_0 \in \mathbf{R}^m$, $\mathrm{v}_0 \in \mathbf{R}^n$. Then, from (16), we have:

$h_0^{\mathrm{T}} \cdot C_j + u_0^{\mathrm{T}} - \mathrm{v}_0^{\mathrm{T}} \cdot Q_j \geq 0$, $j = \overline{2, n}$, hence it follows that:

$$h_0^{\mathrm{T}} \cdot C_j + u_0^{\mathrm{T}} - \mathrm{v}_0^{(j)} \cdot e^{\mathrm{T}} \geq 0,\ j = \overline{2, n} \tag{18}$$

Let $p_0 \in \mathbf{R}^m$, $p_0^{\mathrm{T}} := h_0^{\mathrm{T}} \cdot C_1 + u_0^{\mathrm{T}} - \mathrm{v}_0^{(1)} \cdot e^{\mathrm{T}}$. Then, $p \geq 0$ and from (18) follows that

$$h_0^{\mathrm{T}} \cdot \left(C_j - C_1\right) + p_0^{\mathrm{T}} + \mathrm{v}_0^{(1)} \cdot e^{\mathrm{T}} \geq \mathrm{v}_0^{(j)} \cdot e^{\mathrm{T}},\ j = \overline{2, n} \tag{19}$$

Using (19) with $j = 2$, we have:

$$h_0^{\mathrm{T}} \left(C_2^{[1]} - c_2 \cdot e^{\mathrm{T}}\right) + p_0^{\mathrm{T}} + \left(\mathrm{v}_0^{(1)} + \left(h_0^{\mathrm{T}} \cdot c_2\right)^{+}\right) \cdot e^{\mathrm{T}} \geq \left(\mathrm{v}_0^{(2)} + \left(h_0^{\mathrm{T}} \cdot c_2\right)^{-}\right) \cdot e^{\mathrm{T}} \tag{20}$$

where $t^{+} = \max\{t, 0\}$ and $t^{-} = \max\{-t, 0\}$.

From definition of the matrix D_2 (given in (10)) and of Φ_2 (from (14)), (20) prove that $\left[h_0^{\mathrm{T}}, p_0^{\mathrm{T}}, \mathrm{v}_0^{(1)} + \left(h_0^{\mathrm{T}} \cdot c_2\right)^{+}, \mathrm{v}_0^{(2)} + \left(h_0^{\mathrm{T}} \cdot c_2\right)^{-}\right] \in \Phi_2$.

Now, according to observation 3 and equation (15), there exist $z_2 \in \mathbf{R}^{r_2}$, $w_2 \in \mathbf{R}^m$, and $\gamma_2, \delta_2, \eta_2 \in \mathbb{R}$ satisfying:

$$z_2^{\mathrm{T}} \cdot e = 1,\ \left[z_2^{\mathrm{T}}, w_2^{\mathrm{T}}, \gamma_2, \delta_2, \eta_2\right] \geq 0,$$

$$h_0^{\mathrm{T}} = z_2^{\mathrm{T}} \cdot H_2,\ p_0^{\mathrm{T}} = z_2^{\mathrm{T}} \cdot U_2 + w_2^{\mathrm{T}} + \gamma_2 \cdot e^{\mathrm{T}} \qquad (21)$$

and

$$\mathrm{v}_0^1 + \left(h_0^{\mathrm{T}} \cdot c_2\right)^+ = \delta_2 + \eta_2,\ \mathrm{v}_0^2 + \left(h_0^{\mathrm{T}} \cdot c_2\right)^- = z_2^{\mathrm{T}} \cdot \mathrm{v}_2 + \gamma_2 + \delta_2 \qquad (22)$$

The equations (21) and $p_0^{\mathrm{T}} = h_0^{\mathrm{T}} \cdot C_1 + u_0^{\mathrm{T}} - \mathrm{v}_0^{(1)} \cdot e^{\mathrm{T}}$ prove that $z_2^{\mathrm{T}}\left(H_2 C_1 - U_2\right) = -u_0^{\mathrm{T}} + w_2^{\mathrm{T}} + \left(\mathrm{v}_0^{(1)} + \gamma_2\right) \cdot e^{\mathrm{T}}$, and hence it follows that:

$$z_2^{\mathrm{T}} \cdot H^{[2]} = h_0^{\mathrm{T}}$$

$$z_2^{\mathrm{T}} \cdot U^{[2]} = -u_0^{\mathrm{T}} + w_2^{\mathrm{T}} + \left(\mathrm{v}_0^{(1)} + \gamma_2\right) \cdot e^{\mathrm{T}} \qquad (23)$$

At this moment, with equations (22), replacing h_0^{T} and p_0^{T} from (19), we may have:

$$z_2^{\mathrm{T}} \cdot C_j^{[2]} + w_2^{\mathrm{T}} + \left(\mathrm{v}_0^{(1)} + \gamma_2\right) \cdot e^{\mathrm{T}} \geq \mathrm{v}_0^{(j)} \cdot e^{\mathrm{T}},\ j = \overline{3, n} \qquad (25)$$

Using inequalities (25) for $j = 3$, we have:

$$z_2^{\mathrm{T}} \cdot D_3 + w_2^{\mathrm{T}} + \left(\mathrm{v}_0^{(1)} + \gamma_2 + \left(z_2^{\mathrm{T}} c_3\right)^+\right)^{\mathrm{T}} \geq 0\left(\mathrm{v}_0^{(3)} + \left(z_2^{\mathrm{T}} c_3\right)^-\right) \cdot e^{\mathrm{T}}$$

and, hence:

$$\left[z_2^{\mathrm{T}}, w_2^{\mathrm{T}}, \mathrm{v}_0^{(1)} + \gamma_2 + \left(z_2^{\mathrm{T}} c_3\right)^+, \mathrm{v}_0^{(3)} + \left(z_2^{\mathrm{T}} c_3\right)^-\right] \in \Phi_3.$$

Repeating this reasoning, we establish that exist $\left[z_n^{\mathrm{T}}, w_n^{\mathrm{T}}, \gamma_n, \delta_n, \eta_n\right] \in \mathbf{R}^{r_n + m + 3}$ who satisfy:

$$z_n^{\mathrm{T}} \cdot H = z_n^{\mathrm{T}} \cdot H^{[n]} = h_0^{\mathrm{T}},$$

$$Z_n^{\mathrm{T}} U = z_n^{\mathrm{T}} \cdot U^{[2]} = z_n^{\mathrm{T}} \cdot U^{[n]} = -u_0^{\mathrm{T}} + w_n^{\mathrm{T}} + \left(\mathrm{v}_0^{(1)} + \sum_{j=2}^{n} \gamma_j\right) \cdot e^{\mathrm{T}} \qquad (23^{\mathrm{I}})$$

and

$$z_n^{\mathrm{T}} \cdot V = z_n^{\mathrm{T}} \cdot V^{[n]} =$$

$$= \left[0, \mathrm{v}_0^{(2)} - \left(\mathrm{v}_0^{(1)} + \gamma_2\right) + \eta_2, \mathrm{v}_0^{(3)} - \left(\mathrm{v}_0^{(1)} + \gamma_2 + \gamma_3\right) + \eta_3, \ldots, \mathrm{v}_0^{(n)} - \left(\mathrm{v}_0^{(1)} + \sum_{j=2}^{n} \gamma_j\right) + \eta_n\right] \qquad (24^{\mathrm{I}})$$

Now, since y satisfy $H \cdot y \geq U \cdot a + V \cdot b$ and $z_n \geq 0$, it follows that: $z_n^{\mathrm{T}} \cdot H \cdot y \geq z_n^{\mathrm{T}} \cdot U \cdot a + z_n^{\mathrm{T}} \cdot V \cdot b$. Thus, according to ($23^{\mathrm{I}}$) and ($24^{\mathrm{I}}$), we have:

$$h_0^{\mathrm{T}} \cdot y \geq -u_0^{\mathrm{T}} \cdot a + \mathrm{v}_0^{(1)}\left(e^{\mathrm{T}} \cdot a - \sum_{j=1}^{n} b^{(j)}\right) + \sum_{j=2}^{n} \mathrm{v}_0{}^{(j)} b_j +$$

$$+ w_n^{\mathrm{T}} \cdot a - \sum_{j=2}^{n}\left(\sum_{s=2}^{j} \gamma_s \cdot b^{(j)}\right) + \left(\sum_{j=2}^{n} \gamma_j\right) \cdot e^{\mathrm{T}} \cdot a + \sum_{j=2}^{n} \eta_j \cdot b^{(j)}$$

which imply $h_0^{\mathrm{T}} y \geq -u_0^{\mathrm{T}} a + \mathrm{v}_0^{\mathrm{T}} b$, relation that contradict (17), therefore the demonstration is complete.

Combining propositions 1 and 4, we achieve the most important result:

Theorem 5 The polyhedron $\tilde{Y}$ associate to multiobjective transportation problem (MOTP): Minimize $C \cdot x$ corresponding to $x \in \left\{x \in \mathbb{R}^{mn} \middle| Ax = b, x \geq 0\right\}$, with $d^T = \left[a^T : b^T\right] \in \mathbb{R}^{m+n}$ satisfying $e^{\mathrm{T}} \cdot a = e^{\mathrm{T}} \cdot b$, has the representation $\tilde{Y} = \left\{y \in \mathbb{R}^k \middle| Hy \geq U \cdot a + V \cdot b\right\}$, where $H = H^{[n]}$, $U = U^{[n]}$, $V = V^{[n]}$.

The following observations emphasize that the construction manner of the matrices H, U and V has important implications in applications. In particularly, observation 6 (a)–(c) indicate the fact that once constructed, these matrices may be used even then when some parameters from the problem are changed.

The observation 6 (d) emphasize that in implementation of the algorithm it should be considered priority the order of destinations such that to minimize the necessary calculations to update matrices H, U and V when the costs are fluctuating.

Observation 6

a) Suppose that that the matrices $H = H^{[n]}$, $U = U^{[n]}, V = V^{[n]}$ were constructed. Since these matrices are independent of $d^{\mathrm{T}} = \left[a^{\mathrm{T}} : -b^{\mathrm{T}}\right]$, it follows that if $d_0{}^{\mathrm{T}} = \left[a_0{}^{\mathrm{T}} : -b_0{}^{\mathrm{T}}\right] \in \mathbf{R}^{m+n}$ is any other vector that satisfy $e^{\mathrm{T}} \cdot a_0 = e^{\mathrm{T}} \cdot b_0$, then the polyhedron $\tilde{Y}_0$ associate with **(MOTP$_0$)**: Minimize $C \cdot x$ corresponding to $x \in \left\{x \in \mathbb{R}^{mn} \middle| Ax = b_0, x \geq 0\right\}$ has the representation: $\tilde{Y} = \left\{y \in \mathbb{R}^k \middle| Hy \geq U \cdot a_0 + V \cdot b_0\right\}$

b) Again, suppose that the matrices $H^{[n]}$, $U^{[n]}$ and $V^{[n]}$ were constructed. Moreover, suppose that we add the $n+1$–th destination with the costs matrix C_{n+1}. Then the polyhedron $\tilde{Y}_1$ associate with **(MOTP$_1$)**: Minimize $\left[C : C_{n+1}\right] \cdot x$ corresponding to $x \in \left\{x \in \mathbb{R}^{mn+m} \middle| A_1 x = d_1, x \geq 0\right\}$, where A_1 is $(m+n+1) \times (mn+m)$ transportation restrictions matrix and $d_1^T = \left[a_1^T : -b_1^T\right] \in \mathbb{R}^{m+n+1}$ satisfy $e^{\mathrm{T}} \cdot a_1 = e^{\mathrm{T}} \cdot b_1$ has the representation $\tilde{Y} = \left\{y \in \mathbb{R}^k \middle| H^{[n+1]} y \geq U^{[n+1]} \cdot a_1 + V^{[n+1]} \cdot b_1\right\}$, where $H^{[n+1]}$, $U^{[n+1]}$ and $V^{[n+1]}$ are achieved making an extra algorithm iteration. We mentioned here that the n-th reactualized matrix C_{n+1} can be most easier obtained using relation (6). In particular, $C_{n+1}^{[n]} = H^{[n]} \cdot C_{n+1} - U^{[n]}$.

c) Just like in the cases a) and b), we suppose that $H^{[n]}$, $U^{[n]}$ and $V^{[n]}$ were constructed. If is given an application of availabilities vector $\left[a_2{}^{\mathrm{T}} : -b_2{}^{\mathrm{T}}\right] \in \mathbf{R}^{m+n}$ which satisfy $e^{\mathrm{T}} \cdot a_2 > e^{\mathrm{T}} \cdot b_2$, then the efficiency set $E\left(Y_2\right)$ associated with the problem **(MOTP$_2$)**: Minimize $C \cdot x$ corresponding to

$$x \in \left\{ x = \left(x_1^T, x_2^T, \ldots, x_n^T\right) \in \mathbb{R}^{mn} \middle| \sum_{j=1}^{n} x_j \le a_2, e^T \cdot x_j > b_2^{(j)}, j = \overline{1,n}, x \ge 0 \right\}$$

it is certainly $E\left(\tilde{Y}_1\right)$, where $\tilde{Y}_1$ is the polyhedron associated with (MOTP$_1$) when C_{n+1} is a null $k \times m$ matrix, $a_1^{\mathrm{T}} = a_2^{\mathrm{T}}$ and $d_1{}^{\mathrm{T}} = \left[b_2{}^{\mathrm{T}}, e^{\mathrm{T}} a_2 - e^{\mathrm{T}} b_2\right]$. Just like we specify in b), $\tilde{Y}_1$ may be obtained only by a single iteration of the algorithm performed in addition.

d) We notice that for $1 \le j \le n$, $H^{[i]}$, $U^{[i]}$ and $V^{[i]}$ depends only by the costs matrices $C_1, C_2, ..., C_J$. Hence it follows that is useful to order the destinations by "robustness" decreasing of the costs matrices associated, i.e. destination 1 should be chosen as destination for which the costs are the most little probably to change, while destination n should be chosen as destination for which the costs are the most probably to change.

In particular, we suppose that the cost matrix associated with only one destination is submissively to changes, while the costs matrices associate to all the others destinations will remain fixedly by any by. In this case, it can be arranged such that the destination with variable cost to be treated in the last iteration of the algorithm and, in this way, every time the costs associated with this destination changes. Only one iteration of the algorithm is necessary to achieve the representation of the polyhedron $\tilde{Y}$.

4. EXAMPLE

Consider a multiobjectiv linear transportation problem with $n = m = k = 3$, where the available resources vector is $a^{\mathrm{T}} = [100, 125, 75]$, the application vector is $b^{\mathrm{T}} = [60, 80, 160]$ and the costs matrix is $C = [C_1 : C_2 : C_3]$, where:

$$C_1 = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 5 & 3 \\ 7 & 1 & 5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 6 & -1 \\ 7 & 7 & 7 \end{bmatrix}, \quad C_3 = \begin{bmatrix} -1 & 5 & 4 \\ 3 & 4 & -3 \\ 5 & 1 & 8 \end{bmatrix}.$$

$[H_2 : U_2 : \mathrm{v}_2]$

1	0	1	0	0	0	0
1	0	0	9	9	0	9
0	1	0	0	1	6	6

0	1	0	0	0	5	5
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	1	2	0	0	2
0	0	1	6	0	4	6
$\frac{5}{4}$	$\frac{9}{14}$	0	0	0	0	$\frac{54}{14}$
$\frac{2}{5}$	$\frac{3}{5}$	0	0	$\frac{3}{5}$	0	$\frac{18}{5}$
0	$\frac{1}{4}$	$\frac{3}{4}$	0	0	0	$\frac{3}{2}$
0	$\frac{6}{7}$	$\frac{1}{7}$	0	0	$\frac{34}{7}$	$\frac{36}{7}$
$\frac{4}{13}$	0	$\frac{9}{13}$	$\frac{54}{13}$	0	0	$\frac{54}{13}$
$\frac{34}{97}$	$\frac{54}{97}$	$\frac{9}{97}$	0	0	0	$\frac{327}{97}$

Table 1.

Using these data, we exemplify the algorithm sketched in section 3.

Step 1

Let $j = 2$.

Step 2

Compute $C_j^{[j-1]}$, $C_2^{[1]} = C_2 - C_1 = \begin{bmatrix} -2 & -2 & 7 \\ 2 & 1 & -4 \\ 0 & 0 & 2 \end{bmatrix}$.

Step 3

Compute c_j and D_j. c_j is the infimum of the column vector of the matrix $C_j^{[j-1]}$. Thus, $C_2^{\mathrm{T}} = \left[-2, -4, 0\right]$. The matrix D_j is defined as $D_j = C_j^{[j-1]} - c_j e^{\mathrm{T}}$.

Therefore, $D_2 = \begin{bmatrix} 0 & 0 & 9 \\ 6 & 5 & 0 \\ 0 & 6 & 2 \end{bmatrix}$.

Step 4

Compute $\left[H_j : U_j : \mathrm{v}_j\right]$ by finding all vertices f the polyhedrons Ψ_j defined in (4). Ψ_2 is defined by a four equation (inequalities) system with seven unknowns. It can be verified that there exist just 14 vertices of Ψ_2, and hence it follows that $\left[H_2 : U_2 : \mathrm{v}_2\right]$ is a 14×7 matrix. The rows of this matrix are given in table 1.

Step 5

Compute $H^{[j]}$, $U^{[j]}$ and $V^{[j]}$ according to formulas (6), (7) and (9). For $j=2$, we notice that $H^{[2]} = H_2 \cdot H^{[1]} = H_2 \cdot I = H_2$, $U^{[2]} = H_2 \cdot U^{[1]} - U_2 = H_2 \cdot C_1 - U_2$ and $V^{[2]} = \left[H_2 \cdot V^{[1]} : H_2 c_2 + \mathrm{v}_2 \right] = \left[0, H_2 c_2 + +\mathrm{v}_2 \right]$. The rows of these matrices are given in table 2.

Step 6

If $j = n$, go to **Step 7**. Else, consider $j = j+1$ and go to **Step 2**.

For the given example, we take $j = 2 < 3 = n$, hence we repeat the steps 2–5 for $j=3$ and go to **Step 7**. The columns of $C_3^{[2]}$, c_3 and D_3 are given in table 3. Then, we notice that the system that defines Ψ_3 has 4 equations (inequalities) with 18 unknowns. It can be verify that there exist 119 vertices of Ψ_3, and hence it follows that $\left[H_3 : U_3 : \mathrm{v}_3 \right]$ has 119 rows and 18 columns. With this matrix, it can be compute $H^{[3]}$, $U^{[3]}$ and $V^{[3]}$ according to formulas (6), (7), (9) because of large dimensions we admit the writing of these matrices in an iterative manner.

	$H^{[2]}$			$U^{[2]}$			$V^{[2]}$
1	0	0	3	4	-1	0	-2
1	0	0	-6	-5	-1	0	7
0	1	0	2	4	-3	0	2
0	1	0	2	5	-2	0	1
0	1	0	2	5	3	0	-4
0	0	1	7	1	5	0	0
0	0	1	5	1	5	0	2
0	0	1	1	1	1	0	6
$\frac{5}{14}$	$\frac{9}{14}$	0	$\frac{33}{14}$	$\frac{65}{14}$	$\frac{22}{14}$	0	$\frac{-1}{14}$
$\frac{2}{5}$	$\frac{3}{5}$	0	$\frac{12}{5}$	4	$\frac{7}{5}$	0	$\frac{2}{5}$
0	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{23}{4}$	2	$\frac{18}{4}$	0	$\frac{1}{2}$
0	$\frac{6}{7}$	$\frac{1}{7}$	$\frac{19}{7}$	$\frac{31}{7}$	$-\frac{11}{7}$	0	$\frac{12}{7}$
$\frac{4}{13}$	0	$\frac{9}{13}$	$\frac{21}{13}$	$\frac{25}{13}$	$\frac{41}{13}$	0	$\frac{46}{13}$
$\frac{34}{97}$	$\frac{54}{97}$	$\frac{9}{97}$	$\frac{273}{97}$	$\frac{415}{97}$	$\frac{173}{97}$	0	$\frac{40}{97}$

Table 2.

Step 7

Let $H = H^{[n]}$, $U = U^{[n]}, V = V^{[n]}$. The system $H \cdot y \geq U \cdot a + V \cdot b$ is the wished representation of $\tilde{Y}$. We notice that, this last system may have many redundant inequalities which can be omitted before determinate the efficient structure of $\tilde{Y}$.

In our example, the original inequalities system which defines $\tilde{Y}$ is reduce to a 12 inequalities system given in table 4.

The extreme points set of $\tilde{Y}$ (i.e. the efficient points set of $\tilde{Y}$) can be now determined applying any method from [11], [12], [13] or finding all efficient extreme points of the multiobjective linear transportation problem.

Minimize $I \cdot y$ corresponding to $y \in \tilde{Y}$.

Using ADBASE [14] to solve (MOLP) for the given example, we find the following 7 extreme points:

$$y_1^T = [1225, 670, 1280],\ y_2^T = [1200, 675, 1300],\ y_3^T = [900, 795, 1180]$$
$$y_4^T = [925, 790, 1160],\ y_5^T = [685, 1030, 1160],\ y_6^T = [360, 1095, 1420]$$
$$y_7^T = [285, 1185, 1525]$$

$C^{[2]}$			c_3	D_3		
-4	1	5	-4	0	5	9
5	10	5	5	0	5	0
1	0	6	0	1	0	6
1	-1	5	-1	2	0	6
1	-1	0	-1	2	0	1
-2	0	3	-2	0	2	5
0	0	3	0	0	0	3
4	0	7	0	4	0	7
$\frac{-11}{14}$	$\frac{-4}{14}$	$\frac{25}{14}$	$\frac{-11}{14}$	0	$\frac{7}{14}$	$\frac{36}{14}$
-1	$\frac{2}{5}$	2	-1	0	$\frac{7}{5}$	3
$\frac{-5}{4}$	$\frac{-1}{4}$	$\frac{9}{4}$	$\frac{-5}{4}$	0	1	$\frac{14}{4}$
$\frac{4}{7}$	$\frac{-6}{7}$	$\frac{37}{7}$	$\frac{-6}{7}$	$\frac{10}{7}$	0	$\frac{43}{7}$
$\frac{20}{13}$	$\frac{4}{13}$	$\frac{47}{13}$	$\frac{4}{13}$	$\frac{16}{13}$	0	$\frac{43}{13}$
$\frac{100}{97}$	$\frac{20}{97}$	$\frac{197}{97}$	$\frac{-100}{97}$	0	$\frac{80}{97}$	$\frac{297}{97}$

Table 3.

Noting these extreme points which are obligatory for various inequations, the efficient structure of $\tilde{Y}$, and hence of Y can be determined.

In our example, $E(Y) = c_O\{y_1, y_2, y_3, y_4\} \cup c_O\{y_3, y_4, y_5, y_6\} \cup c_O\{y_6, y_7\}$, where "$c_O$" nominate the convex part.

$y^{(1)} \geq 285$	$y^{(2)} + y^{(3)} \geq 1950$
$y^{(2)} \geq 670$	$2y^{(1)} + 5y^{(2)} \geq 5775$
$y^{(3)} \geq 1160$	$4y^{(1)} + 5y^{(3)} \geq 8540$
$5y^{(1)} + 9y^{(2)} \geq 11655$	$7y^{(1)} + 5y^{(3)} \geq 9620$
$6y^{(1)} + 5y^{(2)} \geq 7635$	$2y^{(1)} + 6y^{(2)} + y^{(3)} \geq 7750$
$y^{(1)} + 5y^{(2)} \geq 4575$	$y^{(1)} + y^{(2)} + y^{(3)} \geq 2875$

Table 4. The inequalities system which define $\tilde{Y}$

5. FINAL CONCLUSIONS

The presented algorithm for constructing matrices H, U and V require and imply a forward dynamic programming approach based on destinations: at destination j, $2 \leq j \leq n$ only the costs associated with destinations $\overline{1, j}$ can be used.

It is important to notice that, the algorithm can be modified such that the iterations rely on sources.

The algorithm will require $m-1$ iterations, and in iteration i only costs associated with sources $1, 2, ..., i$ will be used.

This modified algorithm could be advantageous when $m < n$ or in similar situations to the situations discussed in observation 6 d), but implying changes rather in sources than in destinations.

The main difficulty of the algorithm is that at every iteration is necessary to find all the vertices of a polyhedron. Although there are many algorithms to enumerate the vertices [11], [12], [13], this is an "expensive" calculus. However, according to observation 6, an initial "investment" in constructing matrices H, U and V can be justified if some parameters are submitted to frequently changes.

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